GRAPHS and FUNCTIONS
FUNCTIONS

Terrametra Resources

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2.3
FUNCTIONS

- Relations and Functions
- Domain and Range
- Determining Whether Relations are Functions
- Function Notation
- Increasing, Decreasing and Constant Functions

RELATIONS and FUNCTIONS

RELATIONS and FUNCTIONS

A ***relation*** is a set of ordered pairs.

A ***function*** is a relation in which,
for each distinct value of the first component of the ordered pairs,
there is *exactly one* value of the second component.

Example 1

Deciding Whether Relations Define Functions

1(a) Decide whether the relation defines a function.

$$F=\{(1,2),(-2,4),(3,4)\}$$

Solution:

Relation F *is a function*, because
for each different x-value there is exactly one y-value.
We can show this correspondence as follows …

$$\begin{array}{ccccc} \{\,1, & -2, & 4\,\} & & x\text{-values of } F \\ \downarrow & \downarrow & \downarrow & & \\ \{\,2, & 4, & 4\,\} & & y\text{-values of } F \end{array}$$

Example 1

Deciding Whether Relations Define Functions

1(b) Decide whether the relation defines a function.

$$G=\{(1,1),(1,2),(1,3),(2,3)\}$$

Solution:

As the correspondence below shows, relation G is *not a function* because *one* first component corresponds to *more than one* second component. …

{ 1, 2 } x-values of F

{ 1, 2, 3 } y-values of F

Example 1

Deciding Whether Relations Define Functions

1(c) Decide whether the relation defines a function.

$$H = \{(-4,1),(-2,1),(-2,0)\}$$

Solution:

In relation *H* the last two ordered pairs have the same x-value paired with two different y-values,
so *H* is a relation but *not a function*.

Different y-values

$$H = \{(-4,1),(-2,1),(-2,0)\}$$

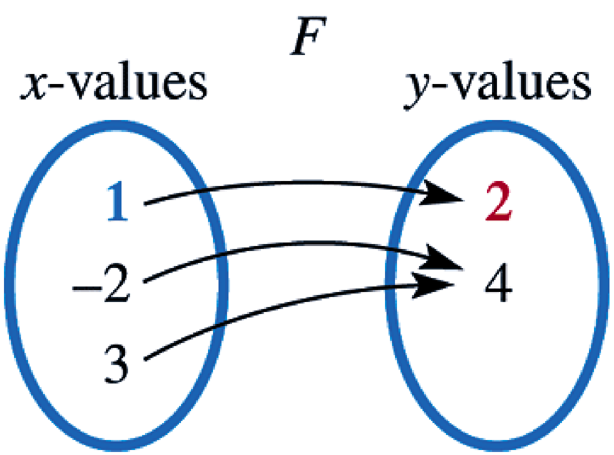
Same x-value

Not a function

Mapping

Relations and functions can also be expressed as a correspondence or *mapping* from one set to another.

In the example below the arrow from 1 to 2 indicates that the ordered pair (1, 2) belongs to *F*. Each first component is paired with exactly one second component.

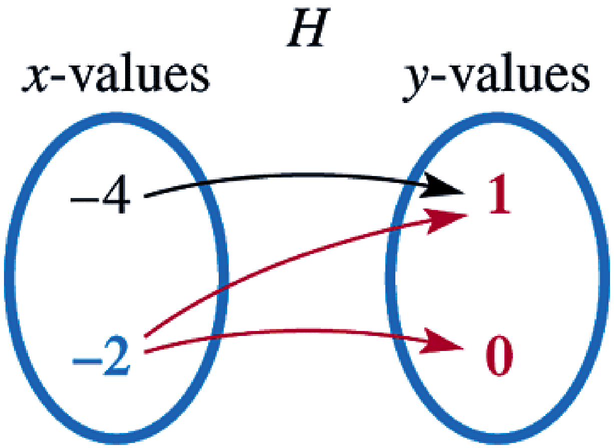


F is a function.

Mapping

Relations and functions can also be expressed as a correspondence or *mapping* from one set to another.

In the mapping for relation *H*, which is not a function, the first component –2 is paired with two different second components, 1 and 0.



H is not a function.

RELATIONS

Note

Another way to think of a function relationship is to think of the independent variable as an input and the dependent variable as an output.

Domain and Range

Domain and Range

In a relation consisting of ordered pairs (x, y) …

The set of all values of the independent variable x is the ***domain*.**

The set of all values of the dependent variable y is the ***range*.**

Example 2

Finding Domains and Ranges of Relations

2(a) Give the domain and range of the relation. Tell whether the relation defines a function.

$$\{(3,-1),(4,2),(4,5),(6,8)\}$$

Solution:

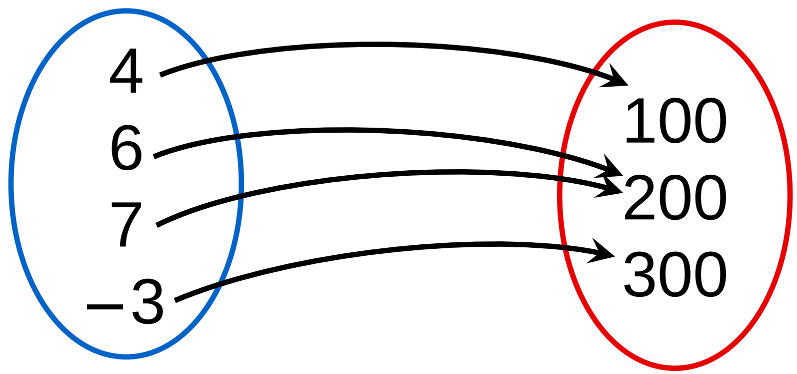
The domain is the set of x-values, {3, 4, 6}.
The range is the set of y-values, {–1, 2, 5, 8}.

This relation is *not a function* because the same x-value, 4, is paired with two different y-values, 2 and 5.

Example 2

Finding Domains and Ranges of Relations

2(b) Give the domain and range of the relation. Tell whether the relation defines a function.



Solution:

The domain is {4, 6, 7, –3}.
The range is {100, 200, 300}.

This mapping *defines a function*.
Each x-value corresponds to exactly one y-value.

Example 2

Finding Domains and Ranges of Relations

2(c) Give the domain and range of the relation.
Tell whether the relation defines a function.

x	y
–5	2
0	2
5	2

Solution:

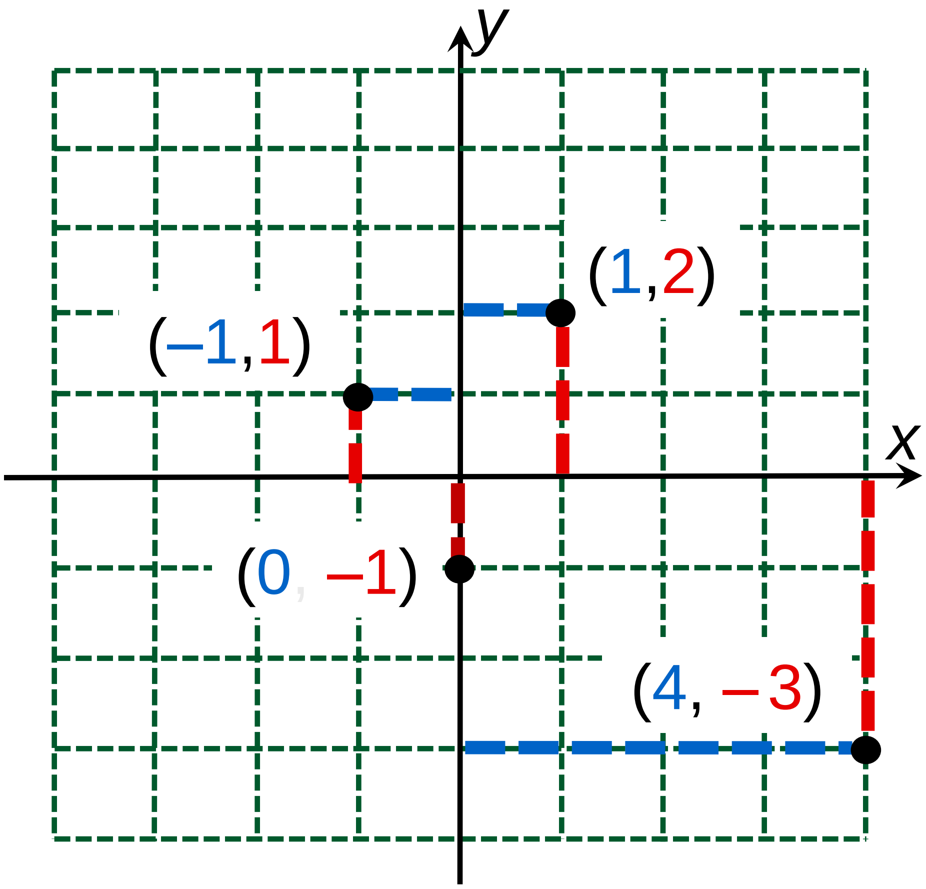
This relation is a set of ordered pairs, so …
the domain is the set of x-values {–5, 0, 5} ;
the range is the set of y-values {2}.

The table *defines a function*
because each different x-value
corresponds to exactly one y-value.

Example 3

Finding Domains and Ranges of Relations from Graphs

3(a) Give the domain and range of the relation.



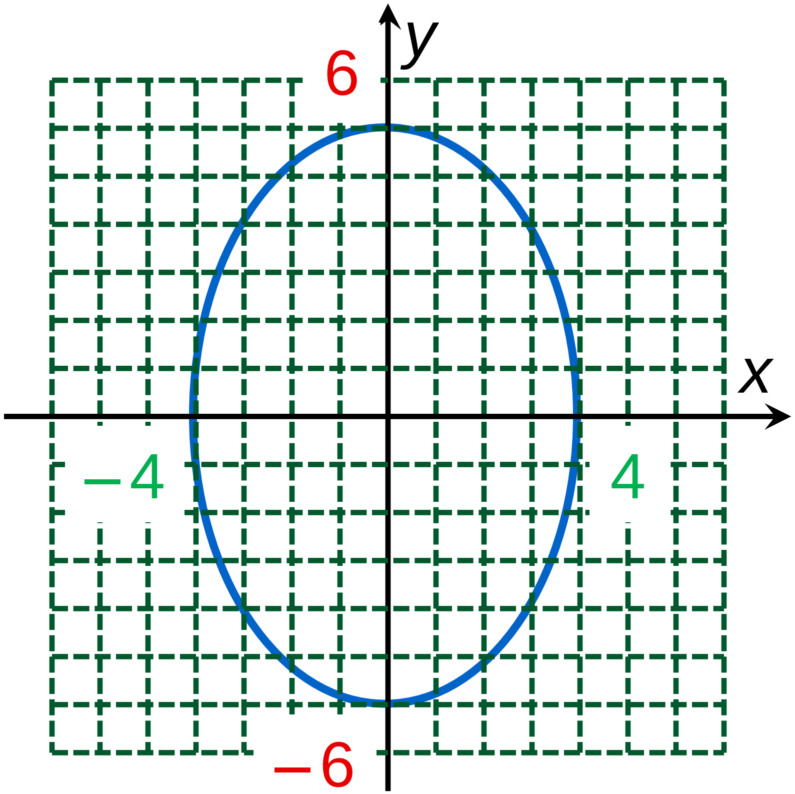
The domain is the set of x-values …
{–1, 0, 1, 4}.

The range is the set of y-values …
{–3, –1, 1, 2}.

Example 3

Finding Domains and Ranges of Relations from Graphs

3(b) Give the domain and range of the relation.



The x-values of the points on the graph include all numbers between –4 and 4, inclusive.

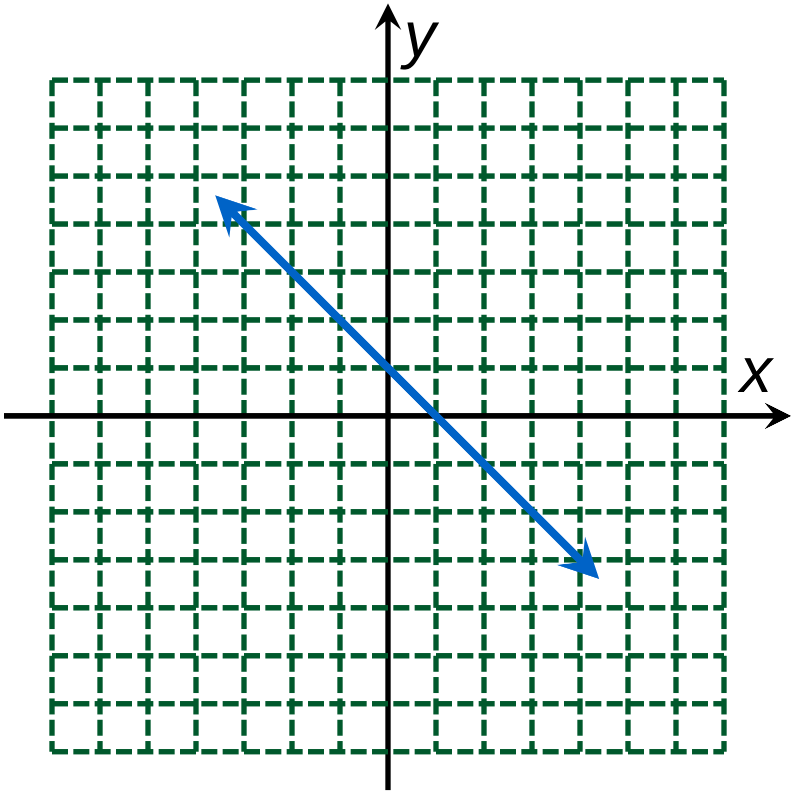
The y-values include all numbers between –6 and 6, inclusive.

The domain is [–4, 4].
The range is [–6, 6].

Example 3

Finding Domains and Ranges of Relations from Graphs

3(c) Give the domain and range of the relation.



The arrowheads indicate that the line extends indefinitely left and right, as well as up and down.

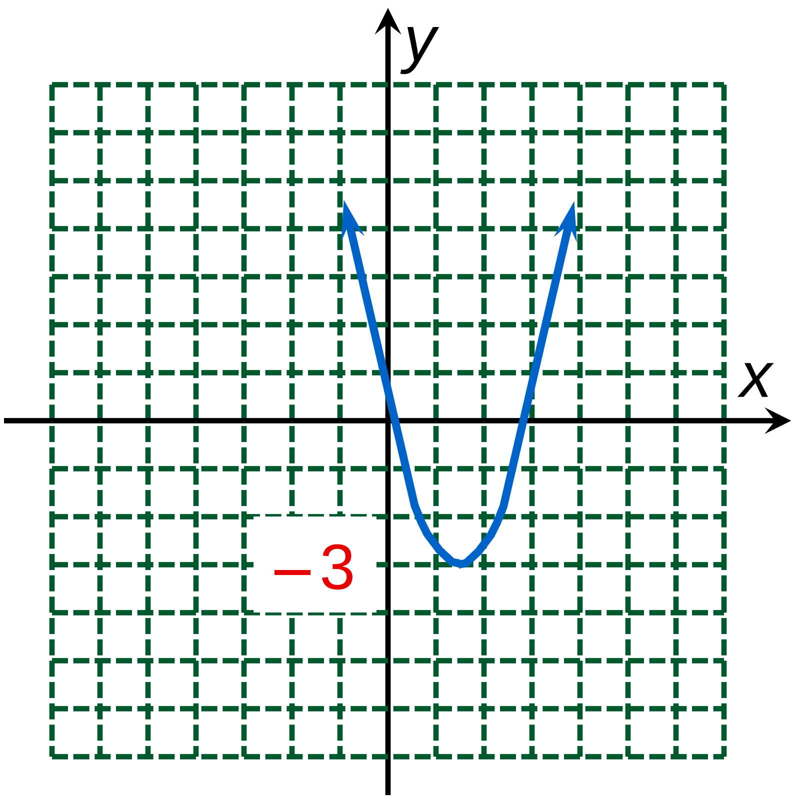
Therefore, both the domain and the range include all real numbers.

The domain is $(-\infty, \infty)$.
The range is $(-\infty, \infty)$.

Example 3

Finding Domains and Ranges of Relations from Graphs

3(d) Give the domain and range of the relation.



The arrowheads indicate that the line extends indefinitely left and right, as well as upward.

Because there is a least y-value, –3, the range includes all numbers greater than or equal to –3.

The domain is $(-\infty, \infty)$.
The range is $[-3, \infty)$.

Agreement on Domain

Argreement on Domain

Unless specified otherwise,
the domain of a relation is assumed to be
all real numbers that produce real numbers
when substituted for the independent variable.

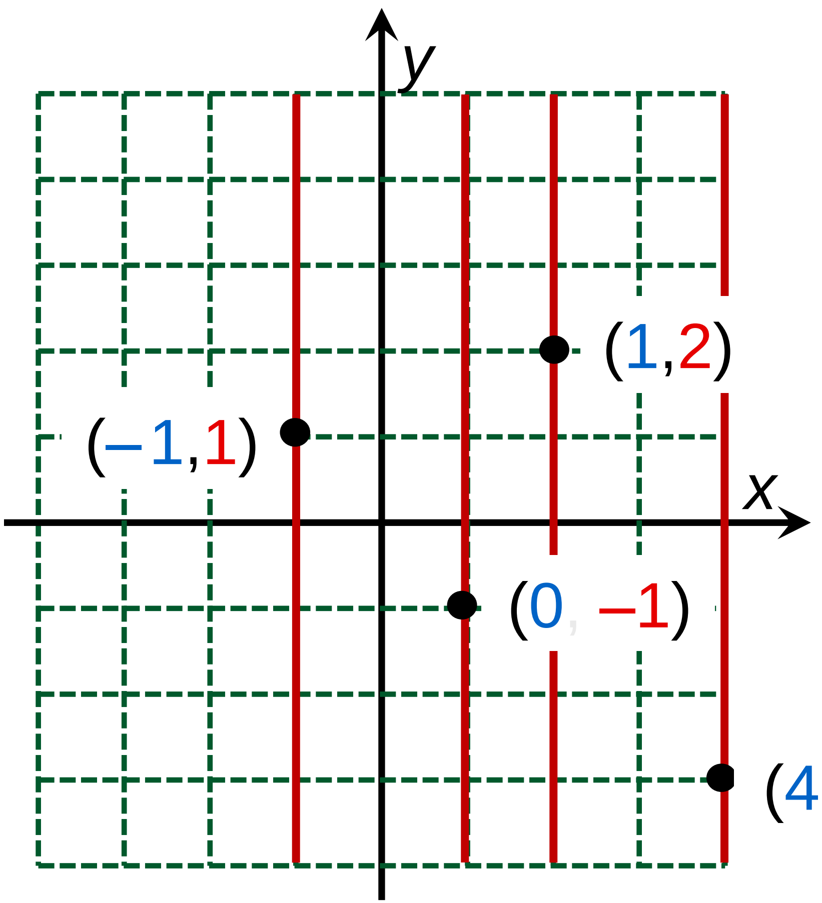
Vertical Line Test

Vertical Line Test

If every vertical line intersects the graph of a relation
in no more than one point,
then the relation is a function.

Example 4
Using the Vertical Line Test

4(a) Use the vertical line test to determine whether the relation graphed is a function.

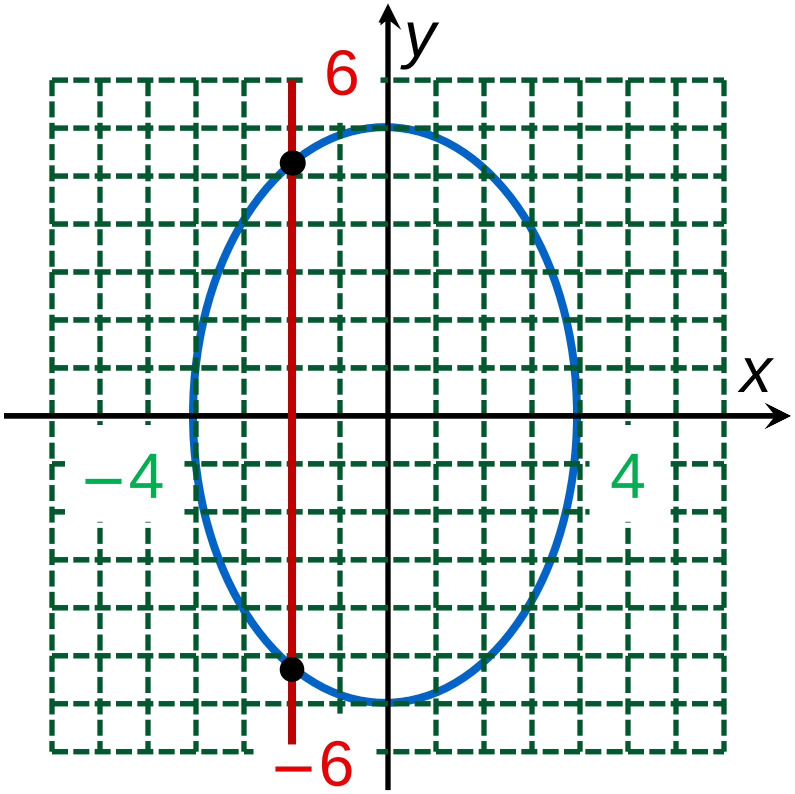


The graph of this relation *passes* the vertical line test, since every vertical line intersects the graph no more than once …

Thus, this graph represents a *function*.

Example 4
Using the Vertical Line Test

4(b) Use the vertical line test to determine whether the relation graphed is a function.



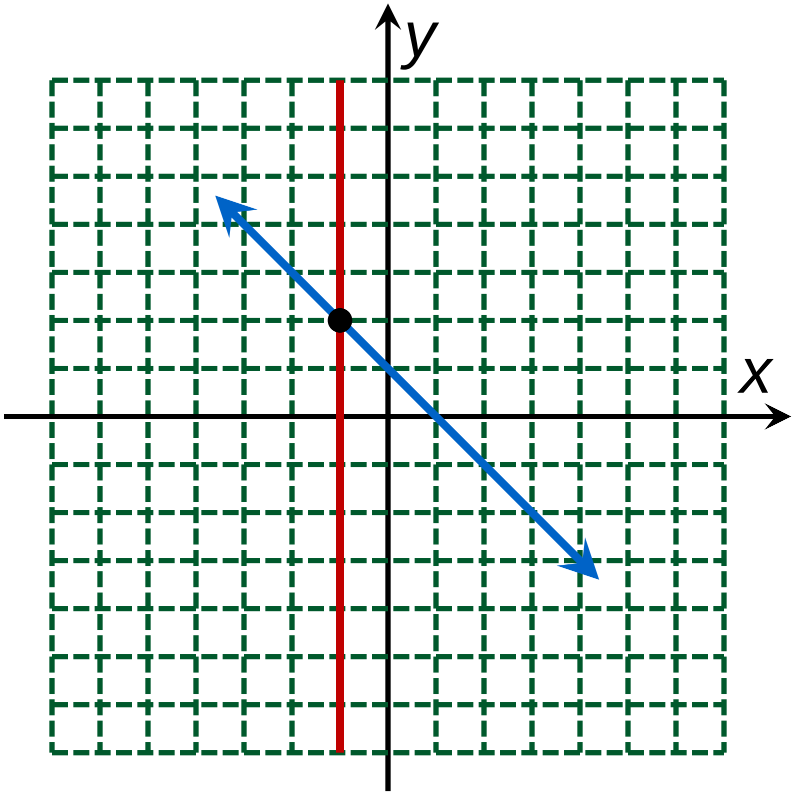
The graph of this relation *fails* the vertical line test, since the same x-value corresponds to two different y-values …

Therefore, it is *not* the graph of a function.

Example 4

Using the Vertical Line Test

4(c) Use the vertical line test to determine whether the relation graphed is a function.

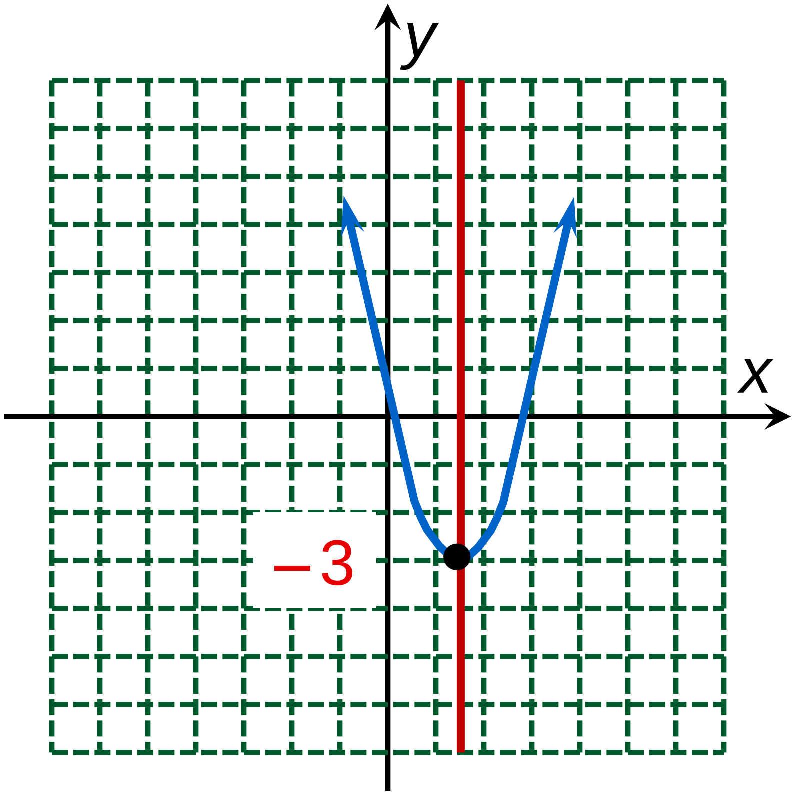


The graph of this relation *passes* the vertical line test, since every vertical line intersects the graph no more than once …

Thus, this graph represents a *function*.

Example 4
Using the Vertical Line Test

4(d) Use the vertical line test to determine whether the relation graphed is a function.



The graph of this relation *passes* the vertical line test, since every vertical line intersects the graph no more than once …

Thus, this graph represents a *function*.

TERRAMETRA

Example 5

Identifying Functions, Domains and Ranges

5(a) Decide whether the relation defines y as a function of x and give the domain and range.

$$\boldsymbol{y = x + 4}$$

Solution:

In the defining equation (or rule), y is always found by adding 4 to x. Thus, each value of x corresponds to just one value of y, and the relation defines a function.

The variable x can represent any real number, so the domain is $\{x|x \text{ is a real number}\}$ or $(-\infty, \infty)$.

Because x is always 4 more than x, y also may be any real number, so the range is also $(-\infty, \infty)$.

Example 5

Identifying Functions, Domains and Ranges

5(b) Decide whether the relation defines y as a function of x and give the domain and range.

$$\boldsymbol{y = \sqrt{2x - 1}}$$

Solution:

For any choice of x in the domain, there is exactly one corresponding value for y (the radical is a nonnegative number), so this equation defines a function. The equation involves a square root, the quantity under the radical cannot be negative …

$$2x - 1 \geq 0 \quad \text{Solve the inequality.}$$

$$2x \geq 1 \quad \text{Add 1.}$$

$$x \geq \frac{1}{2} \quad \text{Divide by 2.}$$

Example 5
Identifying Functions, Domains and Ranges

5(b) Decide whether the relation defines y as a function of x and give the domain and range.

$$\boldsymbol{y = \sqrt{2x - 1}}$$

Solution (cont'd):

$$x \geq \frac{1}{2}$$

Because the radical must represent a non-negative number, as x takes values greater than or equal to $1/2$ …

The domain is $\left[\frac{1}{2}, \infty\right)$.

The range is $\{y|y \geq 0\}$, or $[0, \infty)$.

Example 5
Identifying Functions, Domains and Ranges

5(c) Decide whether the relation defines y as a function of x and give the domain and range.

$$y^2 = x$$

Solution:

The ordered pairs $(16,4)$ and $(16,-4)$ both satisfy the equation. There exists a value of x, 16, that corresponds to two values of y, 4 and -4, so this equation does <u>*not*</u> define a function.

The domain of the relation is $[0, \infty)$.

Any real number can be squared, so …

The range of the relation is $(-\infty, \infty)$.

Example 5

Identifying Functions, Domains and Ranges

5(d) Decide whether the relation defines y as a function of x and give the domain and range.

$$\boldsymbol{y \leq x - 1}$$

Solution:

By definition, y is a function of x if every value of x leads to exactly one value of y. Substituting a particular value of x into the inequality corresponds to many values of y.

The ordered pairs $(1,0)$, $(1,-1)$, $(1,-2)$, and $(1,-3)$ all satisfy the inequality.

Any number can be used for x or for y, so both the domain and the range are the set of real numbers, or $(-\infty, \infty)$.

Example 5
Identifying Functions, Domains and Ranges

5(e) Decide whether the relation defines y as a function of x and give the domain and range.

$$\boldsymbol{y = \frac{5}{x-1}}$$

Solution:

Given any value of x in the domain of the relation, we find y by subtracting 1 from x and then dividing the result into 5. This process produces exactly one value of y for each value in the domain, so this equation defines a function.

Example 5

Identifying Functions, Domains and Ranges

5(e) Decide whether the relation defines y as a function of x and give the domain and range.

$$\boldsymbol{y = \frac{5}{x-1}}$$

Solution (cont'd):

The domain includes all real numbers except those making the denominator 0.

$$x - 1 = 0$$

$$x = 1 \qquad \text{Add } 1.$$

Thus, the domain includes all real numbers except 1 and is written $(-\infty, 1) \cup (1, \infty)$.

The range is the interval $(-\infty, 0) \cup (0, \infty)$.

Variations of the Definition of a Function

Variations of the Definition of a Function

1. A ***<u>function</u>*** is a relation in which, for each distinct value of the first component of the ordered pairs, there is exactly one value of the second component.
2. A ***<u>function</u>*** is a set of ordered pairs in which no first component is repeated.
3. A ***<u>function</u>*** is a rule or correspondence that assigns exactly one range value to each distinct domain value.

Function Notation

When a function f is defined with a rule or an equation using x and y for the independent and dependent variables, we say …

"*y is a function of x*" to emphasize that y <u>*depends*</u> *on* x.

We use the notation $y = f(x)$ called <u>***function notation***</u>, to express this and read $f(x)$ as "f of x" or "f at x" .

The letter f is the name given to this function.

For example, if $y = 3x - 5$,
we can name the function f and write … $f(x) = 3x - 5$.

Function Notation

Note that $f(x)$ is just another name for the dependent variable y.

For example, if $y = f(x) = 3x - 5$ and $x = 2$,

then we find y, or $f(2)$, by replacing x with 2.

$$f(x) = 3 \cdot 2 - 5 = 1$$

The statement "if $x = 2$, then $y = 1$" represents the ordered pair $(2,1)$ and is abbreviated with function notation as …

$$f(2) = 1.$$

The symbol $f(2)$ is read "f of 2" or "f at 2."

Function Notation

These ideas can be illustrated as follows …

Name of the function

$$y = f(x) = \overbrace{3x - 5}^{\text{Defining expression}}$$

Value of the function

Name of the independent variable

RATIONAL INEQUALITIES

Caution

The symbol $f(x)$ does not indicate "f times x,"
but represents
the y-value associated with the indicated x-value.

As just shown ...
$f(x)$ is the y-value that corresponds to the x-value 2.

Example 6

Using Function Notation

6(a) Let $f(x) = -x^2 + 5x - 3$ and $g(x) = 2x + 3$. Find $f(2)$.

Solution:

$$f(x) = -x^2 + 5x - 3$$

$$f(x) = -2^2 + 5 \cdot 2 - 3$$ Replace x with 2.

$$f(2) = -4 + 10 - 3$$ Apply the exponent; multiply.

$$f(2) = 3$$ Add and subtract.

Thus, $f(x) = 3$; the ordered pair $(2,3)$ belongs to f.

Example 6

Using Function Notation

6(b) Let $f(x) = -x^2 + 5x - 3$ and $g(x) = 2x + 3$.
Find $f(q)$.

Solution:

$$f(x) = -x^2 + 5x - 3$$

$$f(x) = -q^2 + 5 \cdot q - 3$$ Replace x with q.

Example 6

Using Function Notation

6(b) Let $f(x) = -x^2 + 5x - 3$ and $g(x) = 2x + 3$.
Find $g(a+1)$.

Solution:

$$g(x) = 2x + 3$$

$$g(a+1) = 2(a+1) + 3$$ Replace x with $a+1$.

$$g(a+1) = 2a + 2 + 3$$ Distribute.

$$g(a+1) = 2a + 5$$ Add.

Example 7

Using Function Notation

7(a) Find $f(3)$.

$$f(x) = 3x - 7$$

Solution:

$$f(x) = 3x - 7$$

$$f(3) = 3 \cdot 3 - 7$$ Replace x with 3.

$$f(3) = 2$$ Simplify.

Thus, $f(3) = 2$; the ordered pair $(3,2)$ belongs to f.

Example 7
Using Function Notation

7(b) Find $f(3)$.

$$f = \{(-3,5),(0,3),(3,1),(6,-1)\}$$

Solution:

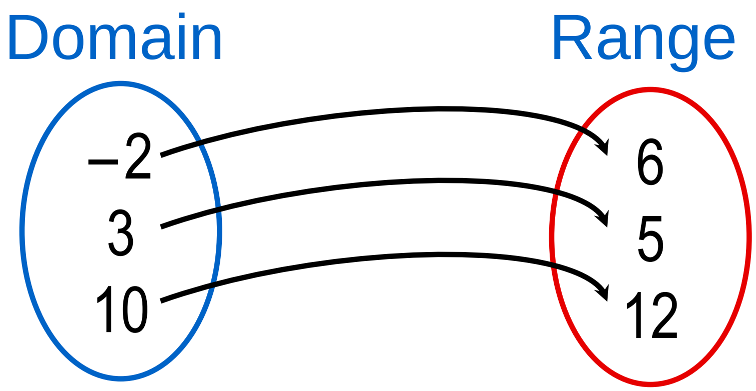
For $f = \{(-3,5),(0,3),(3,1),(6,-1)\}$, we want $f(3)$, the y-value of the ordered pair where $x = 3$.

As indicated by the ordered pair $(3,1)$, when $x = 3$, $y = 1$, so …

$$f(3) = 1.$$

Example 7
Using Function Notation

7(c) Find $f(3)$.



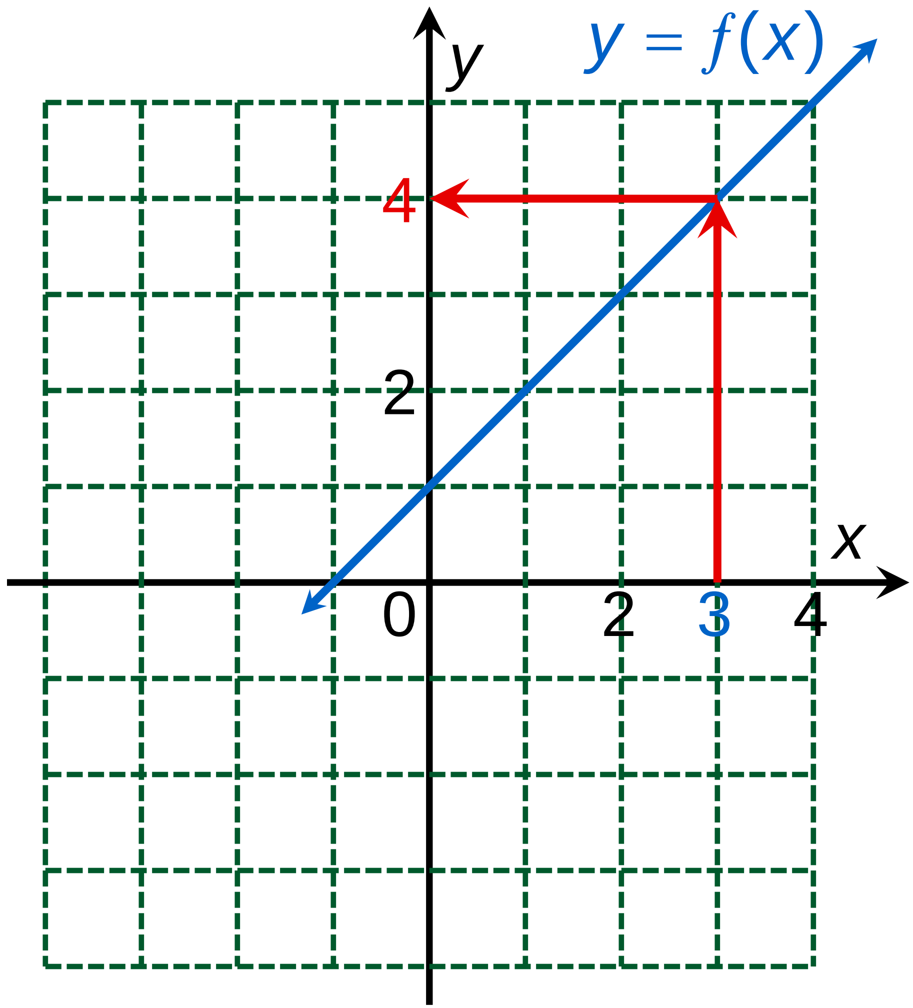
Solution:

In the mapping, the domain element 3 is paired with 5 in the range, so …

$$f(3) = 5.$$

Example 7
Using Function Notation

7(d) Find $f(3)$.



Solution:

Find 3 on the x-axis. Then move up until the graph of f is reached.

Moving horizontally to the y-axis gives 4 for the corresponding y -value.

Thus $f(3) = 4$.

Finding and Expression for $f(x)$

Finding an Expression for $f(x)$

Consider an equation involving *x* and *y*.
Assume that *y* can be expressed as a function f of x.
To find an expression for $f(x)$ use the following steps …

Step 1 Solve the equation for y.

Step 2 Replace y with $f(x)$.

TERRAMETRA

Example 8

Using Function Notation

8(a) Assume that y is a function of x … $\boldsymbol{y = x^2 + 1}$

Rewrite the equation using function notation.

Then find $f(-2)$ and $f(p)$.

Solution:

$$y = x^2 + 1 \qquad \text{Let } y = f(x).$$

$$f(x) = x^2 + 1$$

Now find $f(-2)$ and $f(p)$.

$$f(-2) = (-2)^2 + 1 \qquad \text{Let } x = -2.$$

$$f(-2) = 4 + 1$$

$$f(-2) = 5$$

$$f(p) = p^2 + 1 \qquad \text{Let } x = p.$$

TERRAMETRA

Example 8

Using Function Notation

8(b) Assume that y is a function of x ... $\boldsymbol{x - 4y = 5}$

Rewrite the equation using function notation.

Then find $f(-2)$ and $f(p)$.

Solution:

$$x - 4y = 5$$ Solve for y.

$$-4y = -x + 5$$

$$y = \frac{x-5}{4}$$ Let $y = f(x)$.

$$f(x) = \frac{1}{4}x - \frac{5}{4}$$ $$\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$$

Example 8

Using Function Notation

8(b) Assume that y is a function of x … $\boldsymbol{x - 4y = 5}$

Rewrite the equation using function notation.

Then find $f(-2)$ and $f(p)$.

Solution (cont'd):

$$f(x) = \frac{1}{4}x - \frac{5}{4}$$

Now find $f(-2)$ and $f(p)$.

$$f(-2) = \frac{1}{4}(-2) - \frac{5}{4} = -\frac{7}{4}$$

Let $x = -2$.

$$f(p) = \frac{1}{4}p - \frac{5}{4}$$

Let $x = p$.

Increasing, Decreasing and Constant Functions

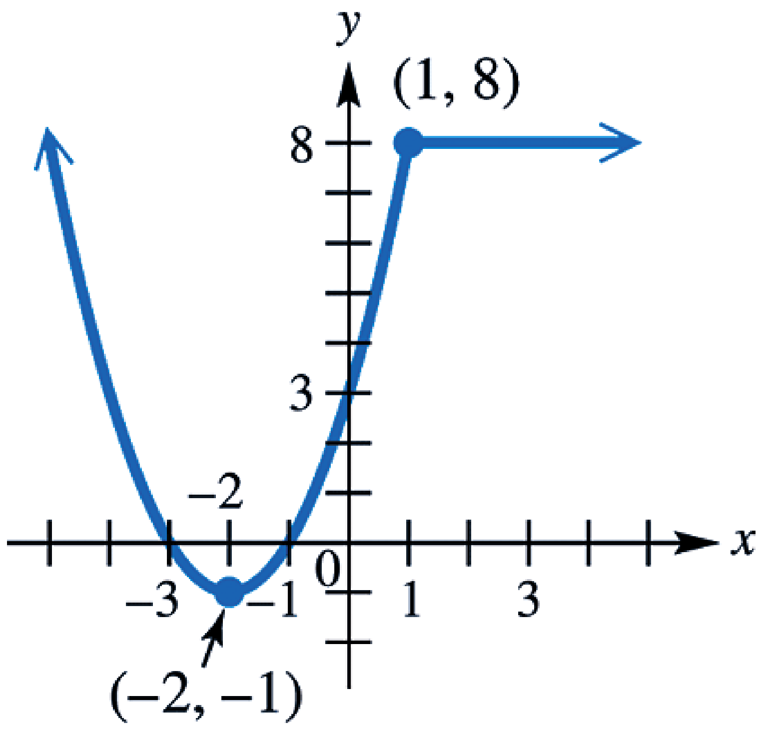
Increasing, Decreasing and Constant Functions

Suppose that a function f is defined over an open interval I and x_1 and x_2 are in I …

(a) f **increases** on I if, whenever $x_1 < x_2$, $f(x_1) < f(x_2)$.
(b) f **decreases** on I if, whenever $x_1 < x_2$, $f(x_1) > f(x_2)$.
(c) f is **constant** on I if, for every x_1 and x_2, $f(x_1) = f(x_2)$.

Example 9
Determining Increasing, Decreasing or Constant Intervals

9(a) Determine the largest open intervals of the domain for which the function is increasing, decreasing, or constant.



Solution:

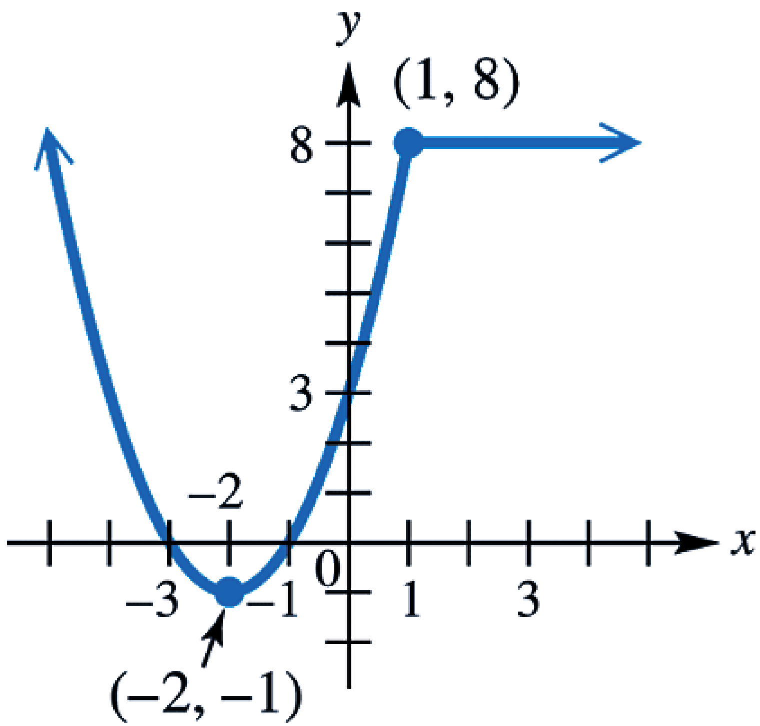
On the open interval $(-\infty, -2)$, the y-values are *decreasing*;

on the open interval $(-2, 1)$, the y-values are *increasing*;

on the open interval $(1, \infty)$, the y-values are *constant* (and equal to 8).

Example 9
Determining Increasing, Decreasing or Constant Intervals

9(a) Determine the largest open intervals of the domain for which the function is increasing, decreasing, or constant.



Solution (cont'd):

Therefore, the function is decreasing on $(-\infty, -2)$, increasing on $(-2, 1)$, and constant on $(1, \infty)$.

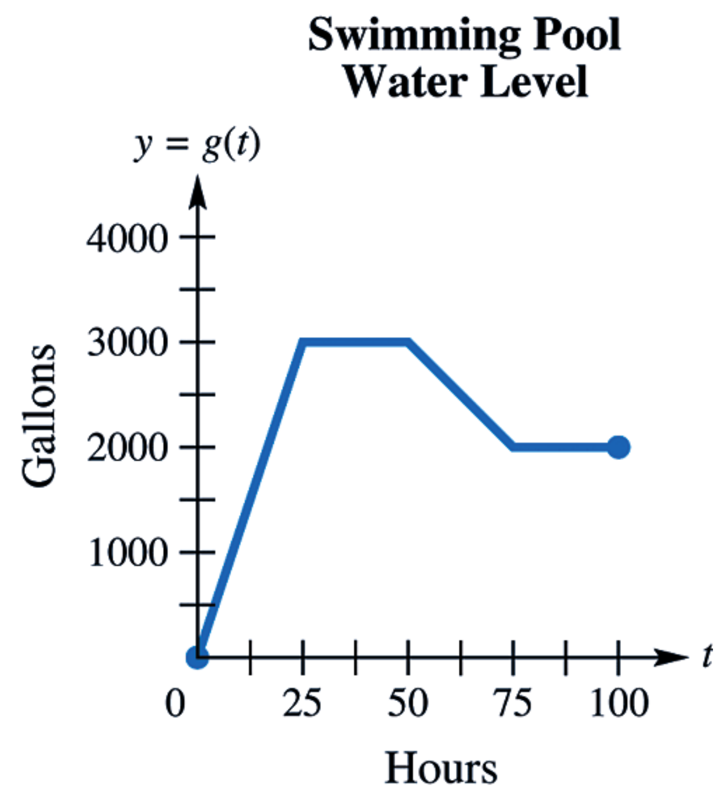
Example 10

Interpreting a Graph

This graph shows the relationship between the number of gallons, $g(t)$, of water in a small swimming pool and time in hours, t.

By looking at this graph of the function, we can answer questions about the water level in the pool at various times …

For example, at time 0 the pool is empty. The water level then increases, stays constant for a while, decreases, and then becomes constant again.



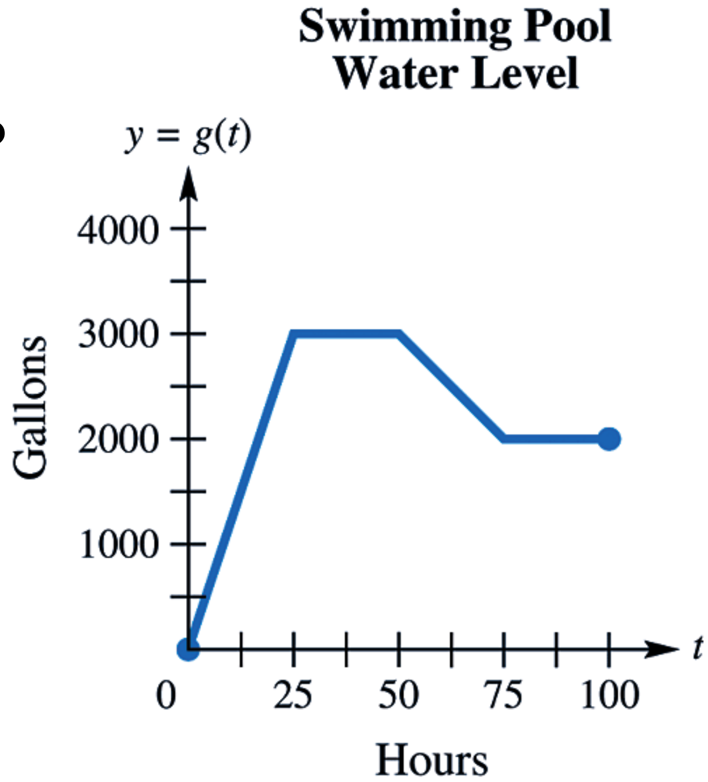
Example 10

Interpreting a Graph

Use the graph to respond to the following …

(a) What is the maximum number of gallons of water in the pool?

(b) When is the maximum water level first reached?



Solution:

The maximum range value is 3000.

This maximum number of gallons, 3000, is first reached at $t = 25$ hr.

Example 10
Interpreting a Graph

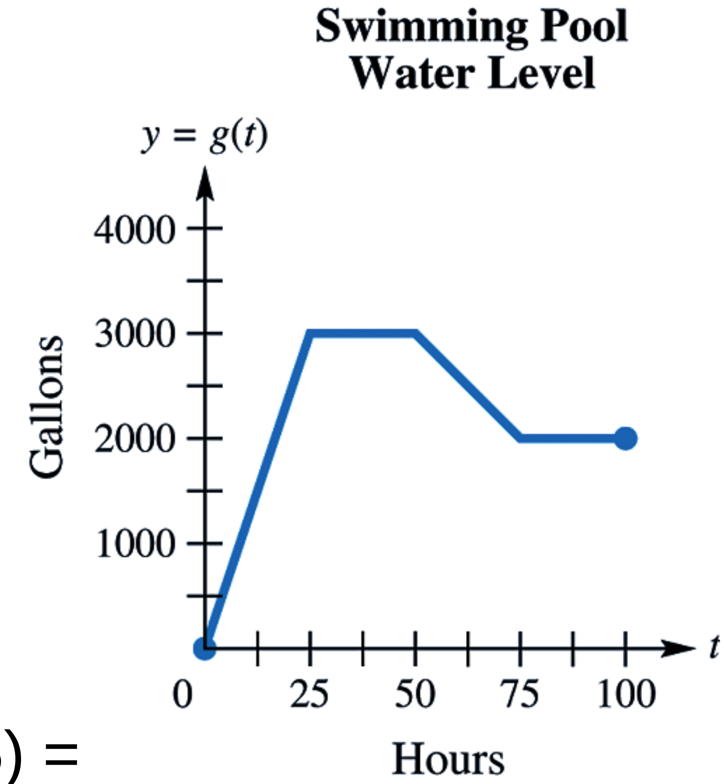
Use the graph to respond to the following …

For how long is the water level …

(c) increasing?

(d) decreasing?

(e) constant?



Solution:

The water level is …

increasing for 25 – 0 = 25 hr.

decreasing for 75 – 50 = 25 hr.

constant for (50 – 25) + (100 – 75) = 25 + 25 = 50 hr.